CHAPTER 7

# Portfolio Evaluation

## Introduction

In the last chapter, we examined how to measure the return and risk of stocks. In an efficient market, investors, however, either individually or through an investment fund, can attain better return-risk opportunities by investing in a portfolio. In this chapter, we analyze portfolios, with the emphasis being on stock portfolios.

Portfolio analysis consists of the evaluation and selection of the financial assets that makeup the portfolio. As with individual security evaluation, stock portfolio evaluation entails measuring the characteristics of the portfolio, with the most important properties being the portfolio expected rate of return and risk. Portfolio selection, in turn, involves finding what proportion of investment funds to allocate to each security to give the portfolio either the maximum expected return for a given risk or the minimum risk given a specified return. In this chapter, we evaluate portfolios of risky stocks in terms of their expected portfolio return and risk, and in Chapter 8, we focus on portfolio selection. We begin by examining the relationship between portfolio return and risk and the relationship between portfolio risk and number of securities in the portfolio. Next, we introduce a risk-free security and show how investors can obtain different return-risk combinations with different allocations of their investment funds to the risk-free security and a portfolio of risky stocks.

## Portfolio Return and Risk

A portfolio can be described by the proportion of investment funds allocated to each security in it. For example, suppose an individual invested $1,000 in three stocks, denoted $X_1$, $X_2$, and $X_3$. Her stock portfolio would be described by the proportion of investment funds ($1,000), denoted as $w_i$, that she allocates to each stock:

$$w_i = \frac{\text{Stock } i \text{ Investment}}{\text{Total Investment}}$$

Thus, if she buys \$200 worth of $X_1$, \$300 of $X_2$, and \$500 of $X_3$, then her portfolio would be described in terms of her proportional allocations of $w_1 = 0.20$, $w_2 = 0.30$, and $w_3 = 0.5$.

In describing a portfolio, two points should be noted. First, the whole must equal the sum of the parts; that is, the proportion of investment funds allocated to each security must sum to one. Thus:

$$\sum_{i=1}^{n} w_i = 1$$

where $n$ = number of stocks in the portfolio.

Second, the allocation or weights, $w_i$, can take on any value. If $w_i < 0$, then there would be a negative investment in security *i*. A negative investment is the opposite of investment, which is borrowing or selling the stock short. In the case of a negative investment in a risky security, such as stock, the negative weight can be interpreted as a short sale in which the investor uses the proceeds from the sale to invest in the other securities in the portfolio. For example, suppose an investor with \$1,000 of investment funds also sells \$500 worth of Stock 1 short (i.e., borrows shares of the stock and sells them in the market for \$500) and uses the proceeds along with his \$1,000 to invest in Stock 2. His portfolio can be described as consisting of two stocks with allocations of $w_1 = -0.5$ and $w_2 = 1.5$:

$$w_1 = \frac{\text{Stock 1 Investment}}{\text{Total Investment}} = \frac{-\$500}{\$1{,}000} = -0.5$$

$$w_1 = \frac{\text{Stock 2 Investment}}{\text{Total Investment}} = \frac{\$1{,}500}{\$1{,}000} = 1.5$$

$$w_1 + w_2 = -0.5 + 1.5 = 1$$

Most stock portfolios constructed by investment companies do not include negative investments or short positions. In our analysis, we will focus on risky stock portfolios with no short positions; that is, all the stocks have positive weights.

## Portfolio Expected Return

The portfolio rate of return is the sum of the weighted rates of return of the securities making up the portfolio, with the weights being the proportion of investment funds allocated to each security:

$$R_p = \sum_{i=1}^{n} w_i r_i = w_1 r_1 + w_2 r_2 + \cdots + w_n r_n \tag{7.1}$$

where:

$R_p$ = portfolio rate of return
$r_i$ = rate of return on security i for the period (holding period yield)

Thus, if the above investor who allocated 20 percent, 30 percent, and 50 percent of her \$1,000 in stocks $X_1$, $X_2$, and $X_3$, attained rates of return of 10 percent, 5 percent, and 15 percent, respectively, for holding these stocks for one year, then her portfolio rate of return (or *HPY*) would be 11 percent:

$$R_p = (0.20)10\% + (0.30)5\% + (0.50)15\% = 11\%$$

Note that in Equation (7.1) the rate of return on the security is defined as a return for the period (total return, *TR*, or holding period yield, *HPY*). Using *HPY* as the measure of return implies an analysis that is static; that is, it applies to only one period in time. The length of time used for portfolio analysis is difficult to determine. Generally, portfolio analysis excludes very short periods (often characterized by speculative trading), and extremely long periods (often subject to greater uncertainty). Static portfolio analysis is therefore restricted to a period between the very short and long run, anywhere between six months and three years. A static analysis of portfolios does not require that the length of time be defined, only that the analysis be constrained to one period instead of multiple periods.

As we noted in our discussion of security return and risk, investors are concerned not with past returns, but with expected returns. The expected portfolio rate of return is the sum of the weighted expected rates of return of the securities making up the portfolio. That is:

$$E(R_p) = \sum_{i=1}^{n} w_i E(r_i) = w_1 E(r_1) + w_2 E(r_2) + \cdots + w_n E(r_n) \tag{7.2}$$

Equation (7.2) is obtained by treating $r_i$ in Equation (7.1) as a random variable and $w_i$ as a constant and then taking the expected value of Equation (7.1) (i.e., apply expected value operator rules).

## Portfolio Risk

Investment risk can be measured by the variance or standard deviation in the security's rate of return. This measure of risk is also appropriate in measuring the risk of a portfolio. From our statistics discussion in Chapter 6, we noted that a portfolio's variance (risk) should depend not only on the variances of the individual stocks that make up the portfolio, but also on the correlation between the stocks composing the portfolio. Recall, the correlation between the rates of return of two securities can be measured by the covariance between their rates or by their correlation coefficient. The covariance is a measure of the extent to which one random variable is above or below its mean at the same time or state that another random variable is above or below its mean. If two random variables, on average, are above their means at the same time, and, on average, below at the same time, then the random variables will be positively correlated with each other and their covariance would be positive. In contrast, if one random variable, on average, is above its mean when another is below, and vice versa, then the random variables would move inversely or negatively to each other and their covariance would be negative.

The covariance between two random variables (e.g., security rates of return), $r_1$ and $r_2$, is equal to the expected value of the product of the variables' deviations. Like any expected value, the covariance can be defined as a weighted sum, with the weights being probabilities associated with each possible product of deviation. That is:

$$Cov(r_1 r_2) = E[r - E(r_1)][r_2 - E(r_2)]$$
$$Cov(r_1 r_2) = \sum_{j=1}^{T} p_j[r_{1j} - E(r_1)][r_{2j} - E(r_2)]$$

Recall from Chapter 6, the correlation coefficient between two random variables such as $r_1$ and $r_2$ ($\rho_{12}$) is equal to the covariance between the variables divided by the product of each random variable's standard deviation, $\sigma$:

$$\rho_{12} = \frac{Cov(r_1 r_2)}{\sigma(r_1)\sigma(r_2)} \tag{7.3}$$

The correlation coefficient has the mathematical property that its value must be within the range of minus and plus one. If two random variables have a correlation coefficient equal to one, they are said to be perfectly positively correlated; if their coefficient is equal to a minus one, then they are perfectly negatively correlated; if their correlation coefficient is equal to zero, then they are uncorrelated and statistically independent:

The covariance or correlation coefficient between two security returns can be estimated using historical averages or a regression model similar to the one described in Chapter 6. An average can be calculated using holding period yields over $N$ historical periods:

$$Cov_{avg} = \frac{1}{N-1}\sum_{t=1}^{N}[HPY_{1t} - \bar{r}_{1t}][HPY_{2t} - \bar{r}_{2t}]$$

The covariance also can be estimated by using a regression model. If we assume two stocks (1 and 2) that are both related to the market, $R^M$, such that

$$r_1 = \alpha_1 + \beta_1 R^M + \varepsilon_1$$
$$r_2 = \alpha_2 + \beta_2 R^M + \varepsilon_2$$

and assume the error terms, $\varepsilon$, are uncorrelated ($Cov(\varepsilon_1\ \varepsilon_2) = 0$), then $Cov(r_1\ r_2)$ simplifies to:

$$Cov(r_1 r_2) = \beta_1 \beta_2 V(R^M)$$

This says that if each stock's unsystematic risk is independent of every other stock's unsystematic risk, then unsystematic risk in the portfolio would average out to zero, and the covariance between any two securities in the portfolio would depend only on each security's systematic risk. A model in which all securities are related just to one factor, such as the market, and their error terms are uncorrelated is referred to as the Single Index Model. It is examined in more detail in the next chapter.

## Importance of the Covariance

The importance of including the covariance between securities' rates of return in measuring portfolio risk can be seen in Exhibit 7.1. The figure is derived from the observations that appear in the exhibit table. The estimated parameters at the bottom of the table are based on the assumption that next period's returns can be obtained from past observations (i.e., on averages). Both the figure and table show the rates of return of two stocks, $X_1$ and $X_2$, over time. Both stocks $X_1$ and $X_2$ have an expected rate of return of 18%, a risk factor as measured by their individual variances of 36%, a covariance of –36, and a correlation coefficient of –1.

An examination of the figure in Exhibit 7.1 shows that when $X_1$'s return ($r_1$) is above its mean, $E(r_1)$, $X_2$'s return ($r_2$) is below its mean, $E(r_2)$, and when $r_2$ is above $E(r_2)$, $r_1$ is below $E(r_1)$; $r_1$ and $r_2$ in this example are perfectly negatively correlated ($\rho_{12} = -1$). If an investor, holding these securities in equal proportion, computed his portfolio rate of return ($R_p$) in time period 3, he would have obtained a rate of return of 18 percent. Similarly, if the investor computed the return for time period 6, he would likewise find an 18 percent rate of return; in fact, with equal weights, the investor would find for any time period that his portfolio rate of return always would be 18 percent. Thus, since the investor can always attain an 18 percent rate of return, there is no portfolio risk. This example therefore illustrates what we first broached in Chapter 6, that the measurement of portfolio risk must take into account not only the risk of each security in the portfolio, but also the correlations that exist between the securities in the portfolio. If we measure the risk of a portfolio by the variance, then both of these factors explicitly are taken into account.

## Derivation of Portfolio Variance Equation

To derive the equation for the variance of portfolio, $V(R_p)$, we start with the definition of $V(R_p)$. That is:

$$V(R_p) = E[R_p - E(R_p)]^2 \tag{7.4}$$

If, for simplicity, we assume a two-security portfolio:

$$R_p = w_1 r_1 + w_2 r_2$$

then the portfolio variance expression (7.4) is

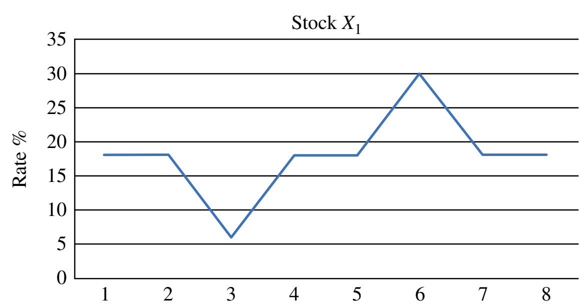
$$V(R_p) = E[w_1 r_1 + w_2 r_2 - w_1 E(r_1) - w_2 E(r_2)]^2 \tag{7.5}$$

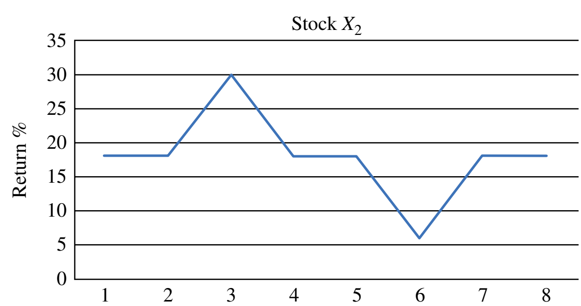
**EXHIBIT 7.1** Correlation between Stock $X_1$ and Stock $X_2$

| Period | Stock $X_1$ Rate of Return, $r_1$ | Stock $X_2$ Rate of Return, $r_2$ |
|---|---|---|
| 1 | 18% | 18% |
| 2 | 18 | 18 |
| 3 | 6 | 30 |
| 4 | 18 | 18 |
| 5 | 18 | 18 |
| 6 | 30 | 6 |
| 7 | 18 | 18 |
| 8 | 18 | 18 |

| $r_{1i}$ | $P_i$ | $P_i\ r_{1i}$ | $P_i\ [r_{1i} - E(r_1)]^2$ | $R_{2i}$ | $P_i$ | $P_i\ r_{2i}$ | $P_i\ [r_{2i} - E(r_2)]^2$ |
|---|---|---|---|---|---|---|---|
| 18% | 6/8 | 13.50% | (6/8) (0) | 18% | 6/8 | 13.50% | (6/8) (0) |
| 6% | 1/8 | 0.75 | (1/8)(144) | 30% | 1/8 | 3.75 | (1/8)(144) |
| 30% | 1/8 | 3.75 | (1/8)(144) | 6% | 1/8 | 0.75 | (1/8)(144) |
| | | 18.00% | $V(r_1)$=36 | | | 18.00% | $V(r_1) = 36$ |
| | | | $\sigma(r_1) = 6$ | | | | $\sigma(r_1) = 6$ |

| $P_i$ | $r_1$ | $R_2$ | $[r_{1i} - E(r_1)]$ | $[r_{2i} - E(r_2)]$ | $P_i\ [r_{1i} - E(r_1)]\ [r_{2i} - E(r_2)]$ |
|---|---|---|---|---|---|
| 6/8 | 18% | 18% | 0 | 0 | $(6/8)(0)(0) = 0$ |
| 1/8 | 6% | 30% | −12 | 12 | $(1/8)(-12)(12) = -18$ |
| 1/8 | 30% | 6% | 12 | −12 | $(1/8)(12)(-12) = -18$ |
| | | | | | $\text{Cov}(r_1 r_2) = -36$ |
| | | | | | $\rho_{12} = -1$ |



Second, we collect the variables in terms of $w_1$ and $w_2$; this yields:

$$V(R_p) = E[w_1[r_1 - E(r_1)] + w_2[r_2 - E(r_2)]]^2 \tag{7.6}$$

Equation (7.6) is similar to $[ab + cd]^2$, which is equal to $a^2b^2 + c^2d^2 + 2abcd$. As a third step, we take the square of Equation (7.6). Similar to $(ab + cd)^2$, this yields:

$$\begin{aligned} V(R_p) = E[&w_1^2[r_1 - E(r_1)]^2 + w_2^2[r_2 - E(r_2)]^2 \\ &+ 2w_1w_2[r_1 - E(r_1)][r_2 - E(r_2)]] \end{aligned} \tag{7.7}$$

The fourth step in the derivation is to apply the expected value operator rules. Applying the rules to Equation (7.7) yields:

$$\begin{aligned} V(R_p) = &w_1^2E[r_1 - E(r_1)]^2 + w_2^2E[r_2 - E(r_2)]^2 \\ &+ 2w_1w_2E[r_1 - E(r_1)][r_2 - E(r_2)] \end{aligned} \tag{7.8}$$

Finally, by definition we know:

$$\begin{aligned} V(r_1) &= E[r_1 - E(r_1)]^2 \\ V(r_2) &= E[r_2 - E(r_2)]^2 \\ Cov(r_1r_2) &= E[r_1 - E(r_1)][r_2 - E(r_2)] \end{aligned}$$

Substituting these expressions into Equation (7.8) yields the desired two-security portfolio variance equation:

$$V(R_p) = w_1^2V(r_1) + w_2^2V(r_2) + 2w_1w_2Cov(r_1r_2) \tag{7.9}$$

The portfolio standard deviation, $\sigma(R_p)$, also can be used as the measure of risk and is obtained simply by taking the square root of (7.9):

$$\sigma(R_p) = \sqrt{V(R_p)} = \sqrt{w_1^2V(r_1) + w_2^2V(r_2) + 2w_1w_2Cov(r_1r_2)} \tag{7.10}$$

Equations (7.9) and (7.10) measure the risk of a two-security portfolio. Note that the portfolio variance includes both the weighted variances of the individual securities' rates of return and the covariance between the securities' returns; hence, the correlation among securities explicitly is taken into account in the equations for the portfolio variance and standard deviation. Moreover, if we substitute into Equation (7.9) the parameter values in Exhibit 7.1 used in the preceding example, we can confirm our graphical interpretation that the risk of that portfolio is indeed zero

$$\begin{aligned} V(R_p) &= w_1^2V(r_1) + w_2^2V(r_2) + 2w_1w_2Cov(r_1r_2) \\ V(R_p) &= (0.5)^2(36) + (0.5)^2(36) + 2(0.5)(0.5)(-36) = 0 \end{aligned}$$

If the securities in the preceding example had not been perfectly negatively correlated, then the portfolio risk would not have been zero. For example, if $X_2$ had a 6% return in time period 3 and a 30% return in period 6, its expected return and variance would still be 18% and 36%. The covariance between $X_1$ and $X_2$, however, would be a positive 36 instead of a negative 36, and therefore $\rho_{12} = +1$ instead of $-1$; the two securities therefore would be perfectly positively correlated in this case. Calculating $V(R_p)$ with the $Cov\ (r_1 r_2) = 36$, we obtain a portfolio variance of 36:

$$V(R_p) = w_1^2 V(r_1) + w_2^2 V(r_2) + 2w_1 w_2 Cov(r_1 r_2)$$
$$V(R_p) = (0.5)^2(36) + (0.5)^2 + 2(0.5)(0.5)(36) = 36$$

Although Equation (7.9) is only for a two-security portfolio, the variance for a larger portfolio necessarily takes the same form, the only difference being the number of inputs (variances and covariance) included. For example, if we have a three-security portfolio, then our variance expression would consist of three security variances and three covariances; that is, the covariances for all combinations between stock returns:

$$\begin{aligned} V(R_p) = {} & w_1^2 V(r_1) + w_2^2 V(r_2) + w_3^2 V(r_3) \\ & + 2w_1 w_2 Cov(r_1 r_2) \\ & + 2w_1 w_3 Cov(r_1 r_3) \\ & + 2w_2 w_2 Cov(r_2 r_3) \end{aligned}$$

If we had a four-security portfolio, then there would be four variances and six covariances, and so on. Equation (7.11) gives a general portfolio variance formula for an $n$-security portfolio.

$$\begin{aligned} V(R_p) = {} & w_1^2 V(r_1) + w_2^2 V(r_2) + \cdots + w_n^2 V(r_n) \\ & + 2w_1 w_2 Cov(r_1 r_2) + \cdots + 2w_1 w_n Cov(r_1 r_n) \\ & + 2w_2 w_3 Cov(r_2 r_3) + \cdots + 2w_2 w_n Cov(r_2 r_n) \\ & + 2w_3 w_4 Cov(r_3 r_4) + \cdots + 2w_3 w_n Cov(r_3 r_n) + \cdots \end{aligned} \tag{7.11}$$

For large portfolios, the number of inputs and thus the size of the portfolio expression can be quite substantial. For example, if we were to compute the variance of a 100-security portfolio, as inputs we would need to compute 100 variances and 4,950 covariances. In general, the number of inputs for any $n$-security portfolio variances is:

- $n$ expected returns.
- $n$ variances.
- $[n^2 - n]/2$ covariances.

### Alternative Portfolio Variance Expressions

The portfolio variance equation also can be expressed in terms of the correlation coefficients by substituting $\rho_{ij}\,\sigma_i\,\sigma_j$ for $Cov(r_i, r_j)$ in Equation (7.11):

$$\begin{aligned} V(R_p) = {} & w_1^2 V(r_1) + w_2^2 V(r_2) + \cdots + w_n^2 V(r_n) \\ & + 2w_1 w_2 \rho_{12} \sigma(r_1)\sigma(r_2) + \cdots + 2w_1 w_n \rho_{1n} \sigma(r_1)\sigma(r_n) \\ & + 2w_2 w_3 \rho_{23} \sigma(r_2)\sigma(r_3) + \cdots + 2w_2 w_n \rho_{2n} \sigma(r_2)\sigma(r_n) \\ & + 2w_3 w_4 \rho_{34} \sigma(r_3)\sigma(r_4) + \cdots + 2w_3 w_n \rho_{3n} \sigma(r_3)\sigma(r_n) + \cdots \end{aligned} \quad (7.12)$$

Equation (7.12) highlights the direct relationship between the degree of correlation among securities in the portfolio and the portfolio's risk: the lower the correlation, the lower the portfolio risk.

The portfolio variance expression can be written compactly with summation signs. If we denote the variance of a security's return as its standard deviation squared and represent it as

$$\sigma(r_i)^2 = \sigma_i^2$$

and we denote the covariance of securities $i$ and $j$ as

$$Cov(r_i r_j) = \sigma_{ij} \quad \text{where} \quad i \neq j$$

then $V(R_p)$ can be expressed as

$$V(R_p) = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} w_i w_j \sigma_{ij} \quad (7.13)$$

Note how the double summation of covariances does yield the two $w_i w_j \sigma_{ij}$ terms. For example, in a three-security case, the double summation term is

$$\begin{aligned} \sum_{i=1}^{3} \sum_{\substack{j=1 \\ j\neq i}}^{3} w_i w_j \sigma_{ij} = {} & w_1 w_2 \sigma_{12} + w_1 w_3 \sigma_{13} + w_2 w_1 \sigma_{21} \\ & + w_2 w_3 \sigma_{23} + w_3 w_1 \sigma_{31} + w_3 w_2 \sigma_{32} \end{aligned}$$

Since, $\sigma_{ij} = \sigma_{ji}$, this expression is

$$\sum_{i=1}^{3} \sum_{\substack{j=1 \\ j\neq i}}^{3} w_i w_j \sigma_{ij} = 2w_1 w_2 \sigma_{12} + 2w_1 w_3 \sigma_{13} + 2w_2 w_3 \sigma_{23}$$

In summary, the portfolio expected return Equation (7.2) and the portfolio variance Equation (7.11) or Equation (7.12) are the important formulas needed to evaluate portfolios. They allow us to quantify any portfolio in terms of its return and risk. By changing the allocations ($w_i$), we change the portfolio's return and risk. The return-risk equations also show that an investor constructing a portfolio based on the criteria of return and risk must search not only for securities with high expected returns and low risks, but also for ones that are uncorrelated and ideally negatively correlated with each other.

## Example

Consider the case of an investor with two stocks, A and B, with the following characteristics:

| | |
|---|---|
| Stock A | $E(r_A) = 10\%$<br>$V(r_A) = 36\%$ |
| Stock B | $E(r_B) = 10\%$<br>$V(r_B) = 36\%$ |
| | $Cov(r_A\ r_B) = 18$<br>$\rho_{AB} = 0.5$ |

Suppose the investor currently has half of her investment funds allocated to each of the stocks, yielding her a portfolio expected rate of return and variance of

$$E(R_p) = (0.5)(10\%) + (0.5)(10\%) = 10\%$$
$$V(R_p) = (0.5)^2(36) + (0.5)^2(36) + 2(0.5)(0.5)(18) = 27$$

Suppose that the investor, however, would like to add a third stock to her portfolio and is considering either stock C, stock D, or stock E, each with the following characteristics:

| | | |
|---|---|---|
| Stock C | $E(r_C) = 10\%$<br>$V(r_C) = 36\%$ | $Cov(r_A\ r_C) = 18$<br>$Cov(r_B\ r_C) = 18$ |
| Stock D | $E(r_D) = 10\%$<br>$V(r_D) = 72$ | $Cov(r_A\ r_D) = 0$<br>$Cov(r_B\ r_D) = 0$ |
| Stock E | $E(r_E) = 10\%$<br>$V(r_E) = 144$ | $Cov(r_A\ r_E) = -36$<br>$Cov(r_B\ r_E) = -36$ |

If she adds stock C to her portfolio and allocates one-third of her investment funds to each stock, her new portfolio expected return and variance would be:

$$E(R_p) = (1/3)(10\%) + (1/3)(10\%) + (1/3)(10\%) = 10\%$$
$$\begin{aligned} V(R_p) = {} & (1/3)^2(36) + (1/3)^2(36) + (1/3)^2(36) \\ & + 2(1/3)(1/3)(18) + 2(1/3)(1/3)(18) \\ & + 2(1/3)(1/3)(18) = 24 \end{aligned}$$

If the investor selects stock D instead of stock C, and again allocates one-third of her funds to each stock, then her portfolio expected return and variance would be:

$$E(R_p) = (1/3)(10\%) + (1/3)(10\%) + (1/3)(10\%) = 10\%$$
$$\begin{aligned} V(R_p) = {} & (1/3)^2(36) + (1/3)^2(36) + (1/3)^2(72) \\ & + 2(1/3)(1/3)(18) + 2(1/3)(1/3)(0) \\ & + 2(1/3)(1/3)(0) = 20 \end{aligned}$$

Finally, if the investor selects stock E as her third security and again uses an equal allocation strategy, then her expected portfolio return and variance would be:

$$E(R_p) = (1/3)(10\%) + (1/3)(10\%) + (1/3)(10\%) = 10\%$$
$$\begin{aligned} V(R_p) = {} & (1/3)^2(36) + (1/3)^2(36) + (1/3)^2(144) \\ & + 2(1/3)(1/3)(18) + 2(1/3)(1/3)(-36) \\ & + 2(1/3)(1/3)(-36) = 12 \end{aligned}$$

Given the choice of adding only one of the stocks to form an equally allocated portfolio, what stock should she choose? If she bases her selection on a portfolio return-risk criterion, stock E would be her best choice. That is, since the same portfolio expected return of 10 percent is attained whether she adds C, D, or E, stock E must be the best since it gives a lower portfolio variance than C or D. What is interesting, however, is that when we measure individual security risk by the variance, stock E is twice as risky as stock D and four times as risky as stock B, yet when we add stock E to the portfolio, it yields the smallest portfolio variance. The reason, of course, is the lower correlation stock E has with stocks A and B compared to the correlations stocks D and E have with A and B:

| Portfolio A, B, C | Portfolio A, B, D | Portfolio A, B, E |
|---|---|---|
| $\rho_{AB} = 0.5$ | $\rho_{AB} = 0.5$ | $\rho_{AB} = 0.5$ |
| $\rho_{AC} = 0.5$ | $\rho_{AD} = 0$ | $\rho_{AE} = -0.5$ |
| $\rho_{BC} = 0.5$ | $\rho_{BD} = 0$ | $\rho_{BE} = -0.5$ |

The example shows the importance of including the correlation in measuring portfolio risk. In addition, if we compare the different portfolio variances obtained with stocks C, D, and E, we confirm our earlier point that the portfolio risk will be lower, the smaller the correlation.

**BLOOMBERG CORRELATION MEASURE SCREENS**

**CORR:** The CORR screen can be used to create and save a number of correlation matrices for securities, indexes, currencies, interest rates, and commodities. As noted in Chapter 6, the matrix also shows a variance-covariance matrix (Cov), correlation coefficient matrix (Correlation), beta, and other correlation and regression parameters.

**PC:** The PC platform shows correlation and regression parameters of a selected stock with its peers and related indexes: S&P 500, Dow Jones, sector indexes, and others. One can select different peers, indexes, portfolios, and securities from searches from the "Bloomberg Peer" dropdown, select different parameters (correlation, beta, covariance, or *t*-statistics) from the "Calculation" tab, different regression periods, and frequencies (e.g., daily, weekly, or monthly).

The PC screen can be saved to the Correlation Matrix from the red "Save to CORR," where it can be accessed for later study by bringing up the CORR screen (CORR <Enter>).

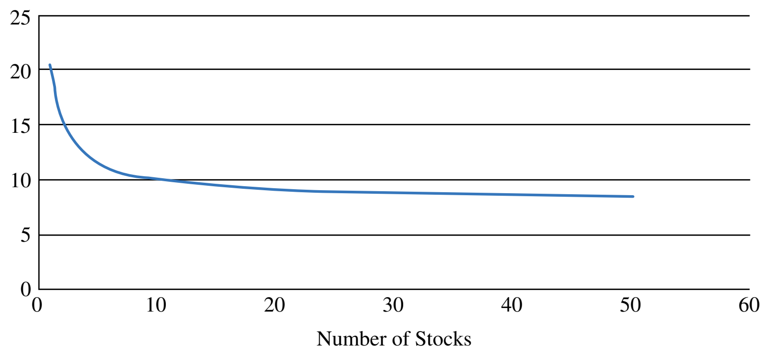
## Portfolio Risk and Size Relation

It can be shown mathematically that as the size of a portfolio increases, the portfolio standard deviation (risk) decreases at a decreasing rate to a point (asymptote) where any additional increase in portfolio size has no impact on portfolio risk. Appendix 7A (text Web site) presents the mathematical derivation of portfolio risk-size relation.

The portfolio risk and size relation has also been examined empirically. Evans and Archer were the first to examine this relationship. In their 1968 study, they calculated the average standard deviations of randomly selected portfolios of different sizes. Specifically, they formed a database of the semi-annual returns of 470 NYSE-listed stocks for the period from 1958 to 1967. Securities were then selected using a random number generator and formed into equally weighted portfolios. They first randomly selected 60 two-stock portfolios and calculated the average standard deviation. Evans and Archer then repeated this process: randomly selecting 60 3-security portfolios, 60 4-security portfolios, and so on up to 60 40-security portfolios, then calculating the average standard deviations for each size portfolio:

$$\bar{\sigma}_n = \frac{1}{60}\sum_{i=1}^{60}\sigma_{ni}, \qquad \text{where: } n = 2, 3, \ldots, 40$$

**EXHIBIT 7.2** Evans and Archer: Risk-Size Relation




The result of the Evans and Archer study are shown in Exhibit 7.2. The graph in the exhibit shows the average portfolio standard deviations that Evans and Archer calculated plotted against their size. This empirically generated portfolio risk and size graph, in turn, shows that as the portfolio size increases, the portfolio risk decreases at a decreasing rate to a point where any additional increase in portfolio size has no impact on portfolio risk. In the Evans and Archer study, the maximum risk reduction is realized with a portfolio of approximately 25 stocks, with the portfolio risk being 0.09.

Recall that in Chapter 6 we defined the unsystematic risk of a security or portfolio as the industry and firm risk that could be diversified away, and we defined a security's or portfolio's systematic risk as the market risk that could not be diversified away. In the context of our discussion here, the Evans and Archer study suggests that a portfolio of approximately 25 to 30 stocks would be needed to eliminate unsystematic risk.

## Return and Risk of a Portfolio of Risky Stocks and a Risk-Free Security

In Chapter 6, we defined a risk-free security as one whose rate of return is known in advance. If we let $R_f$ be the rate of return on the risk-free security, then by definition the variance of $R_f$ is equal to zero: $V(R_f) = 0$. In addition to a zero variance, the risk-free security also is characterized by having a zero covariance with the returns of any other security or portfolio. That is, there is no correlation between a random variable and a constant:

$$Cov(rR_f) = 0$$

Given the opportunity to invest in a risk-free security, investors also want to be able to evaluate a portfolio in which a risk-free security is included with a portfolio of risky stocks. In terms of portfolio evaluation, the inclusion of a risk-free security to a portfolio of risky stocks simply means that we are adding to a stock portfolio a security with a rate of return of $R_f$, a variance of zero, and a covariance with the other securities in the portfolio of zero. However, to highlight the impact of including a riskless security, let us treat the portfolio as a two-security portfolio, with one of the securities being the risk-free one and the other being the portfolio of risky securities with a return of $E(R_p)$ and risk of $V(R_p)$. Denoting $R_I$ as the rate of return on this two-security portfolio, its expected rate of return would be:

$$E(R_I) = w_R R_f + w_p E(R_p) \tag{7.14}$$

where:

$w_R$ = proportion of investment funds allocated to the risk-free security
$w_p$ = proportion of investment funds allocated to the portfolio of risky securities

As with all portfolios, Equation (7.14) is constrained by the condition that the weights sum to one:

$$w_R + w_p = 1$$

Given this constraint, Equation (7.14) can be expressed in terms of just one of the weights, $w_p$, by substituting $1 - w_p$ for $w_R$ in (7.14):

$$E(R_I) = (1 - w_p)R_f + w_p E(R_p)$$
$$E(R_I) = R_f + [E(R_p) - R_f]w_p \tag{7.15}$$

Using the two-security portfolio variance equation, the variance of a portfolio of risky securities and a risk-free security can be expressed as

$$V(R_I) = w_p^2 V(R_p) + w_R^2 V(R_f) + 2w_p w_R Cov(R_f R_p)$$

Given $V(R_f) = 0$ and $Cov(R_f R_p) = 0$, this variance expression simplifies to

$$V(R_I) = w_p^2 V(R_p) \tag{7.16}$$

Finally, taking the square root of (7.16), we obtain the standard deviation of this portfolio:

$$\sigma(R_I) = \sqrt{w_p^2 V(R_I)}$$
$$\sigma(R_I) = \sqrt{w_p^2 \sigma(R_p)^2}$$
$$\sigma(R_I) = w_p \sigma(R_p) \tag{7.17}$$

Equations (7.14) and (7.17) measure the expected return and risk of a portfolio consisting of risky securities and a risk-free security. Note that these equations measure the expected return and standard deviation of returns of the investor's funds. If an investor places all her funds in the portfolio ($w_p = 1$), then her risk would be equal to the portfolio's risk, $\sigma(R_p)$; if she places half of her funds in the risky portfolio and half in the risk-free security, then her investment risk would be equal to half of the portfolio's risk: $(0.5)[\sigma\ (R_p)]$.

## Borrowing and Lending Portfolios

Equations (7.14) and (7.17) can be used to determine the different return-risk opportunities obtainable by changing the allocations of investment funds between the risk-free security and the risky portfolio. For example, suppose an investor is considering investing in a risky stock portfolio with an expected return of $E(R_p) = 10\%$ and a risk (as measured by the standard deviation) of $\sigma(R_p) = 4$. Suppose, however, that the investor is willing to accept a lower expected return for less risk. The investor could change the allocation of the securities in the risky portfolio, with more funds allocated to the less risky stocks. A simpler approach, however, would be to invest only a proportion of funds in the portfolio, the remaining proportion being invested in the risk-free security. For example, suppose the rate on the risk-free security is 5 percent, and the investor decides to place half of his funds in the risk-free security and half in the stock portfolio. Using Equations (7.14) and (7.17), his expected return and risk would be $E(R_I) = 7.5\%$ and $\sigma(R_I) = 2$:

$$E(R_I) = (0.5)(10\%) + (0.5)(5\%) = 7.5\%$$
$$\sigma(R_I) = (0.5)(4) = 2$$

Thus, although the investor's expected return is lower, he has reduced his risk by investing part of his funds in the risk-free security. Moreover, if the investor were still not satisfied with his return-risk opportunity, he could use another allocation strategy. In fact, from a minimum return-risk combination of $E(R_\mathrm{I}) = R_f = 5\%$ and $\sigma(R_I) = 0$, in which $w_R = 1$ to a maximum combination of $E(R_I) = E(R_p) = 10\%$, and $\sigma(R_\mathrm{I}) = \sigma(R_p) = 4$, in which $w_p = 1$, there are an infinite number of return-risk combinations available to the investor. A portfolio consisting of an investment in a portfolio of risky securities and a risk-free security is referred to as a *lending portfolio*. In a lending portfolio, the allocation to the risk-free security is positive, $w_R > 0$, implying an investment in that security.

Recall from our earlier discussion, that the weights can also be negative. As we noted, a negative weight implies a short position. A short position in a risk-free security ($w_R < 0$) could be implemented by borrowing a risk-free security and selling it in the market at a discount rate of $R_f$, then at maturity either buying the bond back at its face value and returning it to the security lender so she can collect the principal or simply paying the principal to her. In this case, the short seller of the risk-free security uses the funds from the short sale to invest in the portfolio of risky securities, and she would pay a rate on those proceeds equal to the risk-free rate. A short position can also be implemented by issuing a risk-free security or by borrowing funds from a financial institution at a rate of $R_f$. In either case, a portfolio of risky securities that is financed in part with funds borrowed at a risk-free rate is referred to as a *borrowing* or *leveraged portfolio*.

In contrast to a lending portfolio, a leveraged portfolio allows an investor to increase his expected return, but at the expense of assuming a higher risk position. For example, suppose the investor in our preceding case wanted an expected return greater than the 10 percent portfolio return and was willing to assume a risk greater than the risky portfolio's risk of $\sigma(R_p) = 4$. The simplest way to accomplish this would be to leverage the portfolio investment by borrowing additional funds to invest in the portfolio. For example, suppose the investor had \$1,000 of his own funds to invest, plus he borrowed an additional \$1,000 at $R_f = 5\%$ to also invest in the stock portfolio. The proportion of his funds allocated to the stock portfolio would therefore be 2, and the proportion of his funds allocated to the risk-free security would be –1 (i.e., negative investment or borrowing):

$$w_p = \frac{\text{Stock Portfolio Investment}}{\text{Total Investment}} = \frac{\$2{,}000}{\$1{,}000} = 2$$

$$w_R = \frac{\text{Borrowed Funds}}{\text{Total Investment}} = \frac{-\$1{,}000}{\$1{,}000} = -1$$

$$w_p + w_R = 2 + (-1) = 1$$

Using Equations (7.14) and (7.17), the expected return and risk of this leveraged portfolio would be $E(R_I) = 15\%$ and $\sigma(R_I) = 8$:

$$E(R_I) = (2)(10\%) + (-1)(5\%) = 15\%$$

$$\sigma(R_I) = (2)(4) = 8$$

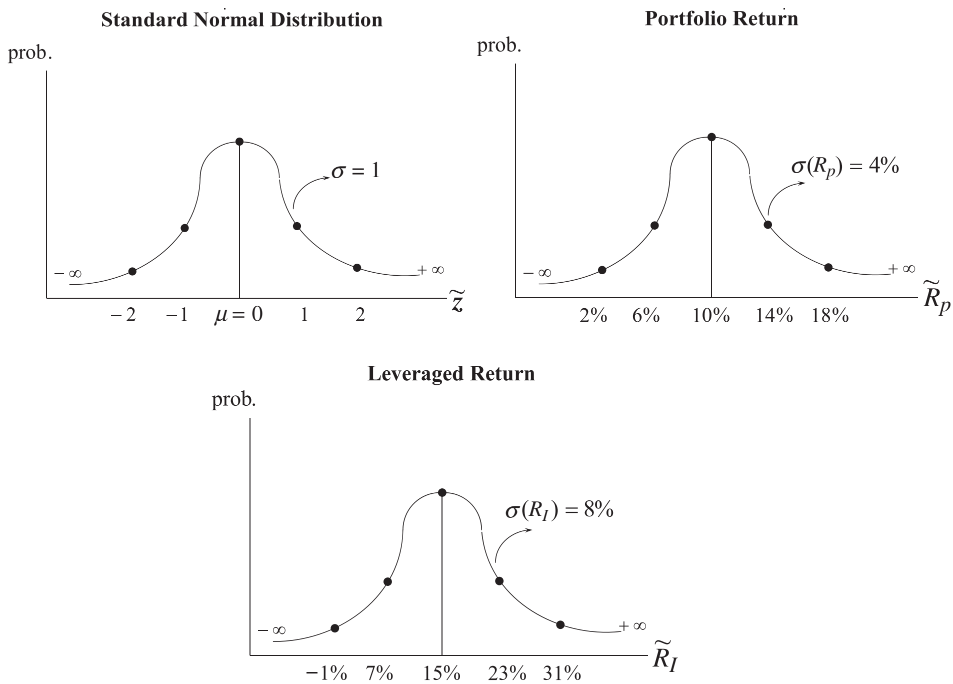
For the investor to actually attain a 15 percent return, the portfolio expected return of 10 percent would have to be realized. If it is, then the investor would receive \$200 from the \$2,000 invested in the stock portfolio, but he would have to pay \$50 on the borrowed funds. His net return would therefore be \$150, yielding a rate of return of 15 percent from his \$1,000 investment, \$150/\$1,000:

$$E(R_I) = \frac{0.10(\$2{,}000) - 0.05(\$1{,}000)}{\$1{,}000} = 0.15$$

The risk of $\sigma(R_p) = 8\%$ suggests that if the portfolio were to yield a return of only 6 percent (i.e., 4 percent less than its expected return or one standard deviation below its mean: $R_p = E(R_p) - \sigma(R_p) = 10\% - 4\% = 6\%$), the investor's return would be 7 percent (i.e., 8 percent less than its expected return or one standard deviation below its mean: $R_I = E(R_I) - \sigma(R_I) = 15\% - 8\% = 7\%$); that is, he would receive only

**EXHIBIT 7.3** Return Distributions of Risk Portfolio and Leveraged Portfolio

| $z$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $R_p = 10 + z4$ | 2% | 6% | 10% | 14% | 18% |
| $R_I = 15 + z8$ | −1% | 7% | 15% | 23% | 31% |




\$120 from portfolio [(0.06)(\$2,000) = \$120] and would have to pay \$50 on the borrowed funds, for a net return of \$70, and a rate of return of 7 percent (\$70/\$1,000):

$$R_I = \frac{0.06(\$2{,}000) - 0.05(\$1{,}000)}{\$1{,}000} = 0.07$$

Note that the assumption that the portfolio's return distribution is normal (i.e., it depends on only the mean and variance) implies that the actual portfolio return of 6 percent and the leveraged return of 7 percent correspond to one deviation below the standard normal distribution, that is, a $z$ score of –1 (see Exhibit 7.3). Thus:

$$R_p = E(R_p) + z\sigma(R_p)$$
$$R_p = 10\% + (-1)(4\%) = 6\%$$

$$R_I = E(R_I) + z\sigma(R_I)$$
$$R_I = 15\% + (-1)(8\%) = 7\%$$

If the actual portfolio return were two standard deviations below the mean, $z\sigma\,(R_p) = (2)(4\%) = 8\%$, then the portfolio return would be 2% (i.e., $R_p = E(R_p) - z\sigma\,(R_p) = 10\% - (2)(4\%) = 2\%$), and the return on the leveraged investment would be –1%:

$$R_I = \frac{0.02(\$2{,}000) - 0.05(\$1{,}000)}{\$1{,}000} = -0.01$$

$$R_I = E(R_I) + z\sigma(R_I)$$
$$R_I = 15\% + (-2)(8\%) = -1\%$$

On the other hand, if the realized portfolio returns were above the expected return, then the returns on the leveraged portfolio will be even higher: For a $z = +1$, $R_p = 14\%$ and $R_I = 23\%$, and for $z = +2$, $R_p = 18\%$, and $R_I = 31\%$. Exhibit 7.3 summarizes the different returns obtained from the portfolio and the leveraged portfolio given different $z$ scores.

In summary, given the opportunity to borrow and lend at a risk-free rate, an investor can attain an unlimited number of return-risk combinations, ranging from a minimum return-risk combination of $R_f$ and zero when $w_R = 1$, to a $E(R_p)$ and $\sigma(R_p)$ combination when $w_p = 1$, to a return-risk combination that is greater than the portfolio's when $w_p > 1$ and $w_R < 0$. Thus, the opportunity to borrow and lend at a risk-free rate, gives the investor a tool for changing his return-risk opportunities without forcing him to change his portfolio's composition of risky securities.

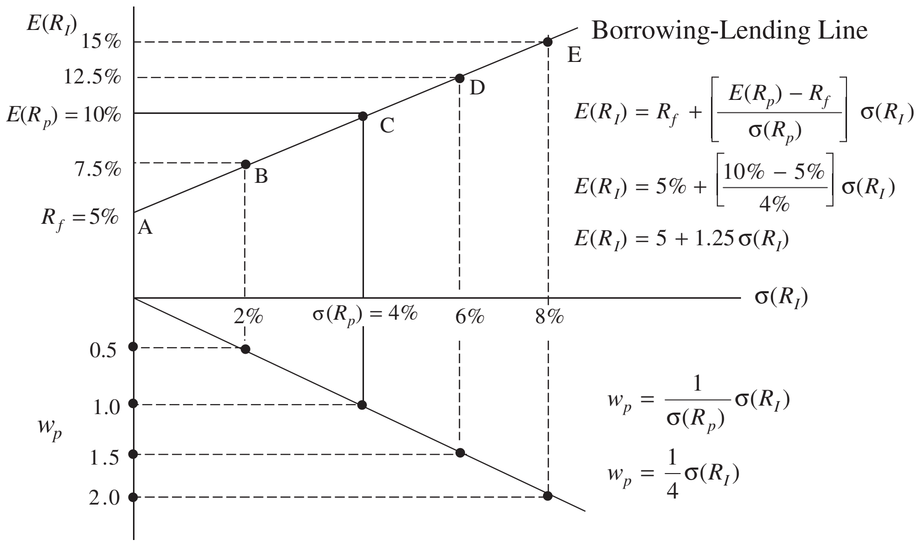
## Return-Risk Relation

Exhibit 7.4 summarizes the return-risk combinations from the preceding example. To reiterate, we assumed our investor could hold a risky portfolio yielding an expected return of 10 percent and a risk of $\sigma(R_p) = 4$, and could borrow and lend at the risk-free rate of 5 percent. Given this portfolio, we generated different return-risk combinations the investor could obtain with different allocations in the risk-free security and the portfolio. Line ABCDE in Exhibit 7.4 is a plot of the return-risk combinations that we obtained. The graph is defined here as the *borrowing-lending line*. The line shows that when the investor places all his funds in the risk-free security, he is on the vertical axis with $\sigma(R_I) = 0$ and $E(R_I) = R_f = 5\%$. By contrast, when the investor places all of his investment funds in the portfolio ($w_p = 1$), he is at point C on the borrowing-lending line where he obtains the portfolio's expected return and risk. Point B lies halfway between points A and C and shows the return-risk combination of $E(R_I) = 7.5\%$ and $\sigma(R_I) = 2$, obtained when the investor places half his investment fund in the portfolio and half in the risk-free security. Finally, point D shows the investor's return-risk combination obtained when he borrows an amount equal to 50 percent above his investment funds ($w_R = -0.5$ and $w_p = 1.5$). The borrowing-lending line, in turn, can be divided into two segments: one segment showing lending portfolios and the other borrowing (or leveraged) portfolios. In particular, the segment to the left of

**EXHIBIT 7.4** Borrowing and Lending Line

| Portfolio | $w_R$ | $w_p$ | $E(R_I)$ | $\sigma\ (R_I)$ |
|---|---|---|---|---|
| A | 1 | 0 | $R_f = 5\%$ | 0 |
| B | 0.5 | 0.5 | 7.5% | 2 |
| C | 0 | 1 | $E(R_p) = 10\%$ | $\sigma\ (R_p) = 4$ |
| D | −0.5 | 1.5 | 12.5% | 6 |
| E | −1 | 2 | 15% | 8 |

**Borrowing-Lending Line**: Graph showing the return-risk combinations obtainable by varying funds between risk-free security and portfolio of risky stocks.




C delineates all the various lending portfolios, whereas the segment to the right of C shows all the borrowing portfolios.

The equation for the borrowing-lending line is

$$E(R_I) = R_f + \left[\frac{E(R_p) - R_f}{\sigma(R_p)}\right] \sigma(R_I)$$

$$E(R_I) = R_f + \lambda\sigma(R_I) \qquad (7.18)$$

where:

$R_f$ = the intercept of the line ABCDE

$\lambda = \left[\frac{E(R_p) - R_f}{\sigma(R_p)}\right]$ is the slope of the line ABCDE

Equation (7.18) is obtained by solving Equations (7.15) and (7.17) simultaneously. Specifically, solving Equation (7.17) for $w_p$ yields

$$w_p = \frac{\sigma(R_I)}{\sigma(R_p)} \tag{7.19}$$

and then substituting $\sigma(R_I)/\sigma(R_p)$ for $w_p$ into Equation (7.15), we obtain Equation (7.18).

Substituting the numerical values of our example for $R_f$, $E(R_p)$, and $\sigma(R_p)$ into Equation (7.18), we obtain the borrowing-lending line equation for ABCDE of

$$E(R_I) = R_f + \left[\frac{E(R_p) - R_f}{\sigma(R_p)}\right]\sigma(R_I)$$
$$E(R_I) = 5\% + \left[\frac{10\% - 5\%}{4\%}\right]\sigma(R_I)$$
$$E(R_I) = 5\% + 1.25\sigma(R_I)$$

Thus, for a specified risk level (e.g., $\sigma(R_I) = 2$), one can determine the expected return [$E(R_I) = 5\% + 1.25\ (2) = 7.5\%$]. In addition, using Equation (7.19), one can also determine the allocation needed to attain the specified risk-return combination [$w_p = \sigma(R_I)/\sigma(R_p) = 2\%/4\% = 0.5$, and $w_R = 1 - w_p = 1 - 0.5 = 0.5$]. The allocations needed to attain the return-risk combinations described by ABCDE are shown by the line in the lower quadrant of the figure in Exhibit 7.4. The line comes from Equation (7.19) and shows $w_p$ plotted against $\sigma(R_I)$, with the slope of the line equal to $1/\sigma(R_p) = 1/4$.

Equation (7.18) is the equation for any borrowing-lending line. It shows the different return-risk combinations that are obtainable by varying one's investment funds between a risk-free security and a given portfolio of risky securities. A special borrowing-lending line is the *Capital Market Line* (*CML*). The CML is formed with the risk-free security and the market portfolio—the portfolio consisting of all securities in the market. Given the market portfolio's expected rate of $E(R^M)$ and its risk of $\sigma(R^M)$, the equation for the CML is

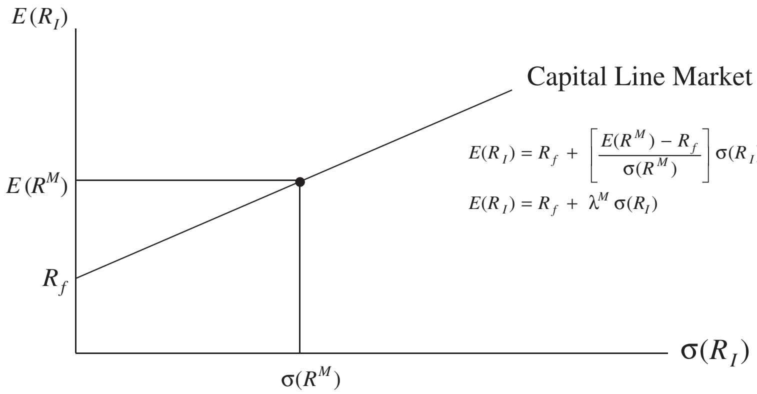
$$E(R_I) = R_f + \left[\frac{E(R^M) - R_f}{\sigma(R^M)}\right]\sigma(R_I)$$
$$E(R_I) = R_f + \lambda^M \sigma(R_I) \tag{7.20}$$

As shown in Exhibit 7.5, the CML shows the different return-risk combinations investors can obtain from different allocations in the risk-free security and the market portfolio.

Before ending our discussion on the return-risk relationship obtained from borrowing and lending portfolios, we should note that throughout the discussion we have

**EXHIBIT 7.5** Capital Market Line

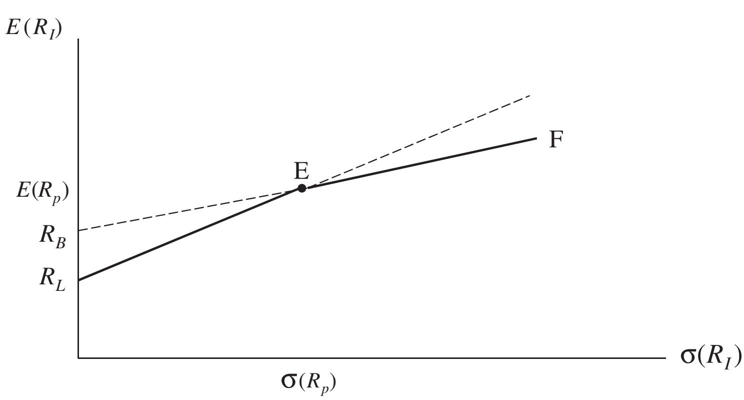
- **Capital Market Line (CML)**: The CML is formed with the risk-free security and the market portfolio. The CML shows the different return-risk combinations investors can obtain from different allocations in the risk-free securities and the market portfolio.




assumed that the risk-free rate that applied to borrowing and lending was the same. Obviously, it is more realistic to assume that the borrowing rate, $R_B$, exceeds the lending rate, $R_L$. If we assume this to be the case, then our borrowing and lending line will no longer be continuous, but rather will exhibit a kink at the return-risk combination where $w_p = 1$; That is, we basically have two lines as shown in Exhibit 7.6, a steeper lending line segment $R_LE$ with a slope of $\lambda_L = [E(R_p) - R_L]/\sigma(R_p)$, and a borrowing segment EF with a slope of $\lambda_B = [E(R_p) - R_B]/\sigma(R_p)$.

**EXHIBIT 7.6** Borrowing and Lending Line with Different Borrowing and Lending Rates

- Borrowing-Lending Line with borrowing rate greater than lending rate. The steeper lending line segment $R_LE$ with a slope of $\lambda_L = [E(R_p) - R_L]/\sigma(R_p)$, and a borrowing segment EF with a slope of $\lambda_B = [E(R_p) - R_B]/\sigma(R_p)$.



## Portfolio Ranking

The return-risk opportunities described by a borrowing-lending line are generated by changing the allocation of investment funds between the risk-free security and the portfolio of risky securities. This return-risk analysis summarized by the borrowing-lending line can be extended to the ranking of different portfolios of risky securities. To see this, suppose the investor in our preceding example was considering an investment in one of two portfolios: portfolio A (the one we just analyzed) or portfolio B. The characteristics of each portfolio are:

| | |
|---|---|
| Portfolio A | $E(R_p) = 10\%$ |
| | $\sigma(R_p) = 4\%$ |
| Portfolio B | $E(R_p) = 20\%$ |
| | $\sigma(R_p) = 8\%$ |

Given the choice of selecting one portfolio, which portfolio should the investor choose?

Because portfolio B yields twice the return and twice the risk as portfolio A, it might appear at first that the investor's decision would depend on his return-risk preference. If he is very risk averse, he would select portfolio A; but if he is less risk averse, then he would select portfolio B. Selecting securities or portfolios based on an investor's risk-return preference is sometimes referred to as the *interior decorator view*. According to this view, an investment advisor would try to match the return-risk characteristic of an investment to the investor's return-risk preferences. In terms of this case, the choice of portfolio would be subjectively based on preference: A timid investor should select portfolio A, and an aggressive investor should select portfolio B.

Given that the investor can borrow and lend at a risk-free rate, however, he is not constrained to the specified return-risk characteristics of the portfolios. That is, he can combine portfolio A with a risk-free security to obtain a set of return-risk combinations (depicted by the borrowing-lending line), as well as combine portfolio B with the riskless security to obtain another set of return-risk combinations (depicted by a different borrowing-lending line). Exhibit 7.7 shows the borrowing-lending lines generated for each of the portfolios given a risk-free rate of 5 percent. The equations for each of the lines are

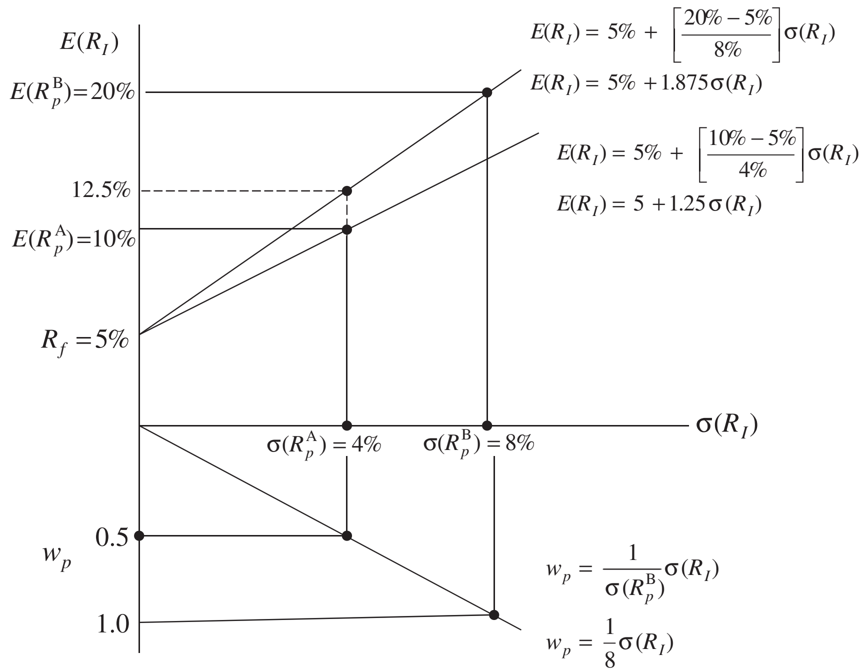
Portfolio A:

$$E(R_I) = 5\% + \left[\frac{10\% - 5\%}{4\%}\right] \sigma(R_I)$$
$$E(R_I) = 5\% + 1.25\sigma(R_I)$$

Portfolio B:

$$E(R_I) = 5\% + \left[\frac{20\% - 5\%}{8\%}\right] \sigma(R_I)$$
$$E(R_I) = 5\% + 1.875\sigma(R_I)$$

**EXHIBIT 7.7** Ranking Portfolios with Borrowing and Lending Lines



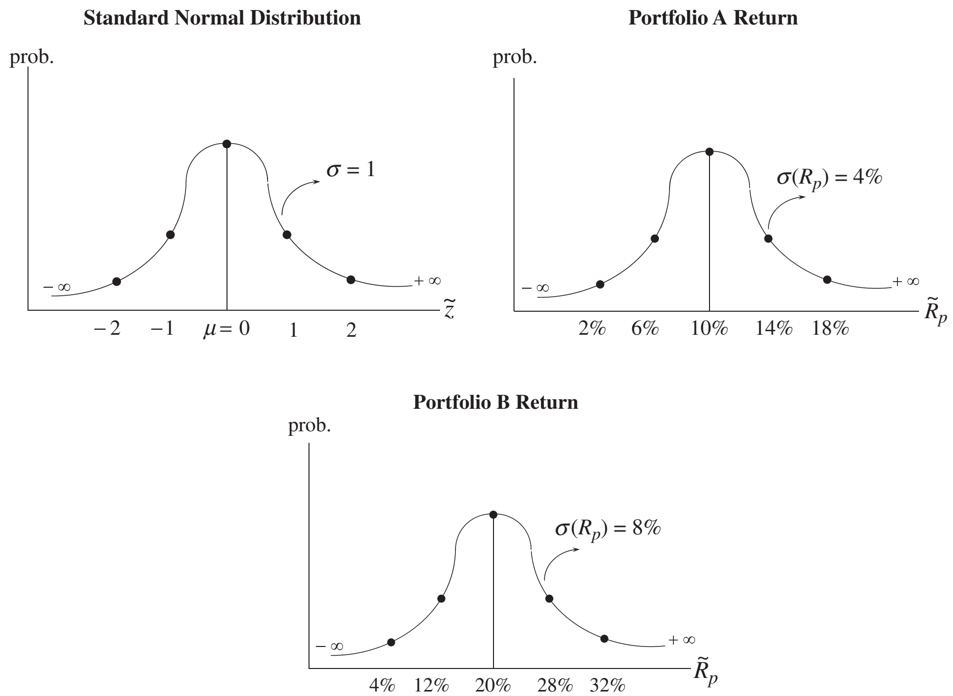

The borrowing-lending line formed with portfolio B is steeper than the line formed with portfolio A; that is, the slope of portfolio B's borrowing-lending line is $\lambda_B = 1.875$, whereas portfolio A's borrowing-lending line slope is $\lambda_A = 1.25$. Given the same intercepts ($R_f$), portfolio B's steeper slope implies that for any risk level, the investor can obtain a greater expected return from an investment in portfolio B and risk-free security than he can from an investment in portfolio A and a risk-free security. For example, suppose the investor was timid, and following the interior decorator view, placed all of his investment funds in portfolio A with the lower expected return and risk of 10 percent and 4 percent. Alternatively, for the same risk of 4 percent, the investor could obtain a higher expected return of 12.5 percent by placing half of his funds in portfolio B and half in the risk-free security:

$$E(R_I) = 5\% + 1.875\sigma(R_I)$$
$$E(R_I) = 5\% + 1.875(4\%) = 12.5\%$$
$$w_p = \frac{\sigma(R_I)}{\sigma(R_p)} = \frac{4\%}{8\%} = 0.5$$

Thus, with the borrowing-lending line formed with B dominating the line formed with A, we can conclude that for any risk, investments formed with portfolio B and the risk-free security yield higher expected returns than those formed with portfolio A and the risk-free security. Portfolio B is therefore the best portfolio.

**EXHIBIT 7.8** Stochastic Dominance

| $z$ | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| A: $R_p = 10 + z4$ | 2% | 6% | 10% | 14% | 18% |
| B: $R_p = 20 + z8$ | 4% | 12% | 20% | 28% | 36‘% |




Mathematically, the only difference between the borrowing-lending lines for portfolio A and B is their slope, $\lambda$. The slope coefficient therefore provides a useful index for ranking portfolios: the greater a portfolio's $\lambda$, the greater its return-risk opportunities, or equivalently, the greater its risk premium, $E(R_p) - R_f$, per level of risk, $\sigma(R_p)$.

The dominance of portfolio B over A also can be seen by comparing its distribution of returns to A's distribution. As shown in Exhibit 7.8, for all $z$-scores covering the distribution, portfolio B's returns are always greater than portfolio A's return. When a random variable has a distribution that dominates another random variable's distribution, it is said to be *stochastically dominant*. Thus, another way to rank portfolios would be to compare their distributions to see which one dominates.

Note that the choice of portfolio B over A is based on an objective criterion of comparing return-risk opportunities or distributions, not on the investor's subjective return-risk preference. This, in turn, suggests that an investor in selecting portfolios (or any asset) based on return and risk should separate the investment decision concerning the choice of portfolios (or assets) from her return-risk preference decision. In finance, the separation of the investment decision from the risk-return preference decision is known as the separation theorem. In terms of our example, the *separation theorem* says that the investor should first select the best portfolio (the one with dominant borrowing-lending line or $\lambda$), regardless of her return-risk preference. Once the best portfolio is determined, the investor can then combine that portfolio with a risk-free security to obtain her desired risk-return preference (i.e., desired point on the borrowing-lending line).

## Portfolio Performance Measures

### *Sharpe Index*

Ranking portfolios by the slopes of their borrowing-lending lines has been a commonly used technique in evaluating the performances of mutual funds, pensions, and other large portfolios. This ranking index is sometimes referred to as the reward-to-variability ratio. The first study to use the slope coefficient to evaluate portfolios was done by William Sharpe. In his study, he evaluated the performance of 34 mutual funds over the decade from 1954 to 1965 using historical return data from *Wiesenberger*. Sharpe calculated each fund's average return and standard deviation and then ranked each in terms of their average risk premium per level of risk using a risk-free rate of 3 percent:

$$\lambda_S = \frac{\bar{R}_p - R_f}{\bar{\sigma}_p}$$

As a standard of comparison, Sharpe also calculated the same ratio for the Dow Jones Industrial Average (DJIA). The data and indexes of the mutual funds are shown in Exhibit 7.9. Sharpe found that the average reward-to-volatility index of the 34 funds was 0.633, which was below the 0.667 index for the DJIA. In fact, of the 34 funds, only 11 had index values higher than the DJIA. Moreover, a study by Lorie and Fisher found that, statistically, the returns from the DJIA were not significantly different from a randomly selected portfolio. Thus, the Sharpe study suggests that many investors during that period would have been better off forming their own portfolios by random selection (e.g., using a random number generator, throwing darts, or having a way for their pets to select stocks) instead of buying mutual fund shares.

### *Treynor Index*

As an index, the slope of the borrowing-lending line ranks portfolios in terms of their risk premium per unit of risk. A variation on this index is to use the portfolio's beta, $\beta_p$, as the measure of risk:

$$\lambda_T = \frac{\bar{R}_p - R_f}{\beta_p}$$

**EXHIBIT 7.9** Sharpe Mutual Fund Ranking

| Mutual Fund | Average Annual Return, % | Standard Deviation of Annual Return, % | Risk Premium* to Standard Deviation Ratio = $S_i$ |
|---|---|---|---|
| Affiliated Fund | 14.6 | 15.3 | 0.75896 |
| American Business Shares | 10.0 | 9.2 | 0.75876 |
| Axe-Houghton, Fund A | 10.5 | 13.5 | 0.55551 |
| Axe-Houghton, Fund B | 12.0 | 16.3 | 0.55183 |
| Axe-Houghton, Stock Fund | 11.9 | 15.6 | 0.56991 |
| Boston Fund | 12.4 | 12.1 | 0.77842 |
| Broad Street Investing | 14.8 | 16.8 | 0.70329 |
| Bullock Fund | 15.7 | 19.3 | 0.65845 |
| Commonwealth Investment Co. | 10.9 | 13.7 | 0.57841 |
| Delaware Fund | 14.4 | 21.4 | 0.53253 |
| Dividend Shares | 14.4 | 15.9 | 0.71807 |
| Dow Jones Industrial Average (DJIA) | 16.3 | 19.9 | 0.66700 |
| Eaton and Howard, Balanced Funds | 11.0 | 11.9 | 0.67399 |
| Eaton and Howard, Stock Fund | 15.2 | 19.2 | 0.63486 |
| Equity Fund | 14.6 | 18.7 | 0.61902 |
| Fidelity Fund | 16.4 | 23.5 | 0.57020 |
| Financial Industrial Fund | 14.5 | 23.0 | 0.49971 |
| Fundamental Investors | 16.0 | 21.7 | 0.59894 |
| Group Securities, Common Stock Fund | 15.1 | 19.1 | 0.63316 |
| Group Securities, Fully Administered Fund | 11.4 | 14.1 | 0.59490 |
| Incorporated Investors | 14.0 | 25.5 | 0.43116 |
| Investment Company of America | 17.4 | 21.8 | 0.66169 |
| Investors Mutual | 11.3 | 12.5 | 0.66451 |
| Loomis-Sales Mutual Fund | 10.0 | 10.4 | 0.67358 |
| Massachusetts Investors Trust | 16.2 | 20.8 | 0.63398 |
| Massachusetts Investors - Growth Stock | 18.6 | 22.7 | 0.68687 |
| National Investors Corporation | 18.3 | 19.9 | 0.76798 |
| National Securities - Income Series | 12.4 | 17.8 | 0.52950 |
| New England Fund | 10.4 | 10.2 | 0.72703 |
| Putnam Fund of Boston | 13.1 | 16.0 | 0.63222 |

**EXHIBIT 7.9** (*Continued*)

| Mutual Fund | Average Annual Return, % | Standard Deviation of Annual Return, % | Risk Premium* to Standard Deviation Ratio = $S_i$ |
|---|---|---|---|
| Scudder, Stevens & Clark Balanced Fund | 10.7 | 13.3 | 0.57893 |
| Selected American Shares | 14.4 | 19.4 | 0.58788 |
| United Funds - Income Funds | 16.1 | 20.9 | 0.62698 |
| Wellington Fund | 11.3 | 12.0 | 0.69057 |
| Wisconsin Fund | 13.8 | 16.9 | 0.64091 |

*$S_i$ = (average return − 3 percent)/std. dev. The ratios shown were computed from original data and thus differ slightly from ratios obtained from the rounded data shown in the table.
*Source:* William F. Sharpe 1966. Mutual fund performances. *Journal of Business* (January, Suppl.): 125.

Recall that in Chapter 6 we defined beta as a measure of systematic risk. Thus, a portfolio with a beta of two would have twice the fluctuations as the market and would be considered to have twice the systematic risk as a portfolio with a beta of one. A portfolio's beta can be estimated by regressing the portfolio's return against the market:

$$R_p = \alpha_p + \beta_p R^M + \varepsilon_p$$

Ranking portfolios in terms of their risk premium per level of systematic risk was first introduced by Treynor in a study of how to rate investment funds. Accordingly, the index is often referred to as the Treynor index.

### *Jensen Index*

A third way of ranking portfolios is to estimate a portfolio's risk-adjusted return. This can be done by first regressing a portfolio's risk premium ($R_p - R_f$) against the market risk premium ($R^M - R_f$):

$$R_p - R_f = \alpha_p + \beta_p[R^M - R_f] + \varepsilon_p$$

From the regression, the intercept term, $\alpha_p$, can be used to measure the portfolio's risk-adjusted return. If the intercept is positive, for example, it suggests the portfolio is generating a return in excess of the risk premium. Using the intercept as a portfolio performance measure was first introduced by Jensen and is often referred to as the Jensen index:

$$\lambda_J = \alpha_p$$

Because the intercept measures only a portfolio's risk-adjusted return, to be used as a reward-to-risk ranking measure in evaluating different portfolios, it needs to be divided by the portfolio risk (as measured by either the portfolio beta or standard deviation).

## Portfolio Performance Evaluation Using Bloomberg

The historical returns and risk of a portfolio can be analyzed using Bloomberg's Portfolio Risk & Analytics Screen, PORT. Exhibit 7.10 shows several PORT performance screens for the Xavier Student Investment Fund (XSIF). The top screen in the exhibit shows the total return of the fund relative to the Russell 1000 index (RIY). The Russell 1000 index consists of the largest 1,000 companies of the Russell 3000. For the period from 9/6/2006 to 9/6/2013, the XSIF fund outperformed the Russell with a total return of 68.30 percent compared to 51.17 percent for the index. The lower screen in Exhibit 7.10 also shows for the seven-year period a lower annualized standard deviation of 25.92 for the XSIF fund compared to 27.80 for the Russell (last two columns). With the higher return and lower risk, the XSIF fund has a larger Sharpe index, 0.50, than the index, 0.40. Note, however, that over the more recent periods of six months and YTD, the XSIF underperformed the Russell and had a lower Sharpe index. Over the three-month period, however, the fund rebounded, outperforming the Russell 1000 with higher returns, lower standard deviations, and a higher Sharpe index.

In addition to performance, Bloomberg's PORT screen can also be used to analyze other features of a portfolio, such as the portfolio composition, sector breakdowns, and attributes. Bloomberg portfolio construction using the PRTU screen and portfolio analysis using screens on the PMEN menu and PORT are described in Chapter 2 and in the Bloomberg exhibit box in this chapter: "Bloomberg Portfolio Screens for Evaluating Portfolios: PRTU and PORT." The box in this chapter and the Bloomberg screens in Bloomberg Web Exhibit 7.1 found on the text Web site show more of the features of the XSIF fund.

## Conclusion

In this chapter, we started our analysis of portfolios by examining how to evaluate portfolios of risky securities in terms of their expected returns and risk. In contrast to individual securities, a portfolio's risk characteristics depend on the correlations of the security returns. If two securities are perfectly negatively correlated, a portfolio can be constructed with zero risk. We then introduced a risk-free security and showed how investors can obtain different return-risk opportunities by varying their investment funds between the risk-free security and a portfolio of risky stocks. In the next chapter, we continue our analysis of portfolios by examining Markowitz portfolio selection for determining the allocation of securities making up the portfolio.

**EXHIBIT 7.10** Return Performances of the XSIF Fund and the Russell 1000—Bloomberg PORT Screen


(a)

<HELP> for explanation.

Port XSIF Equity vs RUSSELL 1000 I by GICS Sectors in USD — As Of 09/06/13

Unit Percentage

| Portfolio Statistics | 3 Months Port | 3 Months Bench | 6 Months Port | 6 Months Bench | Year To Date Port | Year To Date Bench | 7 Year(s) Port | 7 Year(s) Bench |
|---|---|---|---|---|---|---|---|---|
| 5. Return | | | | | | | | |
| Total Return | 4.13 | 3.06 | 7.43 | 8.84 | 13.36 | 18.37 | 68.30 | 51.17 |
| Maximum Return | 1.61 | 1.51 | 1.61 | 1.51 | 2.39 | 2.54 | 11.69 | 11.67 |
| Minimum Return | -2.41 | -2.51 | -2.41 | -2.51 | -2.41 | -2.51 | -8.61 | -9.11 |
| Mean Return (Annualized) | 26.09 | 19.29 | 22.90 | 27.61 | 30.17 | 42.36 | 14.75 | 12.89 |
| Mean Excess Return (Annualized) | 5.70 | | -3.70 | | -8.57 | | 1.65 | |
| 6. Risk | | | | | | | | |
| Standard Deviation (Annualized) | 12.66 | 14.13 | 12.65 | 13.95 | 12.70 | 13.70 | 25.92 | 27.80 |
| Downside Risk (Annualized) | 9.83 | 10.77 | 9.72 | 10.75 | 9.54 | 10.36 | 18.79 | 20.20 |
| Skewness | -0.90 | -0.68 | -0.75 | -0.75 | -0.49 | -0.55 | -0.06 | -0.10 |
| VaR 95% (ex-post) | -1.16 | -1.34 | -1.11 | -1.37 | -1.13 | -1.29 | -2.04 | -2.26 |
| Tracking Error (Annualized) | 4.32 | | 4.14 | | 4.36 | | 5.06 | |
| 7. Risk/Return | | | | | | | | |
| Sharpe Ratio | 2.05 | 1.36 | 1.80 | 1.97 | 2.36 | 3.08 | 0.50 | 0.40 |
| Jensen Alpha | 9.58 | | -1.05 | | -7.09 | | 2.78 | |
| Information Ratio | 1.32 | | -0.89 | | -1.97 | | 0.33 | |
| Treynor Measure | 0.30 | | 0.26 | | 0.34 | | 0.14 | |
| Beta (ex-post) | 0.85 | | 0.87 | | 0.88 | | 0.92 | |
| Correlation | 0.9538 | | 0.9561 | | 0.9483 | | 0.9847 | |

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(b)